MANIFOLDS COVERED BY LINES, DEFECTIVE MANIFOLDS AND A RESTRICTED HARTSHORNE CONJECTURE

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ABSTRACT. Small codimensional embedded manifolds defined by equations of small degree are Fano and covered by lines. They are complete intersections exactly when the variety of lines through a general point is so and has the right codimension. This allows us to prove the Hartshorne Conjecture for manifolds defined by quadratic equations and to obtain the list of such Hartshorne manifolds. Using the geometry of the variety of lines through a general point, we characterize scrolls among dual defective manifolds. This leads to an optimal bound for the dual defect, which improves results due to Ein. We discuss our conjecture that every dual defective manifold with cyclic Picard group should also be secant defective, of a very special type, namely a local quadratic entry locus variety.

INTRODUCTION

The present paper is a natural sequel to our previous work, see [Ru, IR, IR2] and also [BI]. The geometry of the variety of lines passing through the general point of an embedded projective manifold is investigated, in the framework of Mori Theory. Two, rather different, applications are obtained. First, scrolls are characterized among all dual defective manifolds, and the latter are related to some special secant defective ones. Secondly, the quadratic case of the Hartshorne Conjecture is proved, and all extremal examples other than complete intersections are described.

We consider n-dimensional irreducible non-degenerate complex projective manifolds $X \subset \mathbb{P}^{n+c}$. We call X a *prime Fano manifold of index* $i(X)$ if its Picard group is generated by the hyperplane section class H and $-K_X = i(X)H$ for some positive integer $i(X)$. One consequence of Mori's work [Mo] is that, for $i(X) > \frac{n+1}{2}$, X is *covered by lines*, i.e. through each point of X there passes a line, contained in X. As noticed classically, Fano *complete intersections* with $i(X) \geqslant 2$ are also covered by lines. The "biregular part" of Mori Theory (no singularities, no flips, ...), see [Mo2, De, Ko], provides the natural setting for studying manifolds covered by lines. For instance, as first noticed in [BSW], when the dimension of the variety of lines passing through a general point is at least $\frac{n-1}{2}$, there is a Mori contraction of the covering family of lines. Moreover, its general fiber (which is

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still covered by lines) has cyclic Picard group, thus being a prime Fano manifold. For prime Fanos, the study of covering families of lines is nothing but the classical aspect in the theory of the *variety of minimal rational tangents*, developed by Hwang and Mok in a remarkable series of papers, see e.g. [HM, HM2, HM3, Hw]. We also recall that lines contained in X play a key role in the proof of an important result due to Barth–Van de Ven and Hartshorne, cf. [Ba]. It states that X must be a complete intersection when its dimension is greater than a suitable (quadratic) function of its degree.

Prime Fanos of high index other than complete intersections are quite rare. For instance, *dual defective* manifolds and some special but important *secant defective* ones provide such examples. Thus, when $\mathrm{Pic}(X) = \mathbb{Z}\langle H\rangle$, the class of *local quadratic entry locus* varieties, see [Ru, IR, IR2] gives examples with $i(X) = \frac{n+\delta}{2}$, δ being the secant defect, while dual defective manifolds have $i(X) = \frac{n+k+2}{2}$, where k is the dual defect, see [E2].

Another intriguing fact is that all known examples of prime Fanos of high index are either complete intersections or *quadratic*, i.e. scheme theoretically defined by quadratic equations. Mumford, [Mum], was the first to call the attention to the fact that many special but highly interesting embedded manifolds are quadratic. One crucial remark we made is that prime Fanos of high index are embedded with *small codimension*, which naturally leads us to the famous *Hartshorne Conjecture*, see [Ha]: if $n \geqslant 2c+1$, X should be a complete intersection. This conjecture is already very difficult even in the special case of (prime) Fano manifolds. Early contributions related to the conjecture came by with a topological flavor. First the celebrated Fulton–Hansen Connectivity Theorem [FH], followed by Zak's Linear Normality Theorem and his equally famous Theorem on Tangencies, see [Za]. Then, the beautiful result of Faltings [Fa], later improved by Netsvetaev [Ne], showing that X is a complete intersection when the *number* of equations scheme theoretically defining it is small. Another development, suggested by classical work of Severi, is due to Bertram–Ein–Lazarsfeld [BEL]. They characterize complete intersections in terms of the *degrees* of the first c equations defining X (in decreasing order).

Consider now $x \in X$ a general point and let $\mathcal{L}_x$ denote the variety of lines through x, contained in X. Note that $\mathcal{L}_x$ is naturally embedded in $\mathbb{P}^{n-1}$ = the space of tangent directions at x. Let $a := \dim(\mathcal{L}_x)$ and note that for prime Fanos, $i(X) = a+2$. When $a \geqslant \frac{n-1}{2}$, $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is smooth, irreducible and, due to a key result by Hwang [Hw], non-degenerate. Our Theorem 2.4, inspired by [BEL, Corollary 4], shows that we can relate the equations of $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ to those of $X \subset \mathbb{P}^N$. In particular, manifolds of small codimension and defined by equations of small degree are covered by lines; moreover, $X \subset \mathbb{P}^N$ is a complete intersection if and only if $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is so and its codimension is the right one. In case X is quadratic, we combine the above with Faltings' Criterion [Fa] to deduce that, when $n \geqslant 2c+1$, $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a complete intersection. Next, we appeal to a different ingredient, leading to a proof of the Hartshorne Conjecture for quadratic manifolds, see Theorem 4.8. The necessary new piece of information is provided by the

projective second fundamental form which, due to the Fulton–Hansen Connectivity Theorem, turns out to be of maximal dimension. We would like to point out that, working with local differential geometric methods, in the spirit of [GH, IL], Landsberg in [La] proved that when $n \geqslant 3c+b-1$ a (possibly singular) quadratic variety X is a complete intersection, where $b = \dim(\mathrm{Sing}(X))$.

Using Netsvetaev's Theorem [Ne], we show that the only quadratic manifolds $X \subset \mathbb{P}^{\frac{3n}{2}}$ which are not complete intersections are $\mathbb{G}(1,4) \subset \mathbb{P}^9$ and the spinorial manifold $S^{10} \subset \mathbb{P}^{15}$, see our Theorem 4.9.

In a somehow opposite direction, starting from an observation due to Buch [Bu], we show that for a manifold covered by lines, when $a \geqslant n-c$ each line from the covering family is part of the contact locus of a suitable hyperplane. This property never holds for complete intersections. We elaborate on this and prove that under some mild condition (which is, conjecturally, automatically fulfilled) for dual defective manifolds with cyclic Picard group each general line is a contact line. This allows us, using also [BFS], to characterize scrolls among all dual defective manifolds, see Theorem 3.6. As a consequence, we get an optimal bound on the dual defect, generalizing one of the main results in Ein's foundational papers [E2, E1]. His other main result is recovered, with a different proof. We also give evidence for our conjecture asserting that dual defective manifolds with cyclic Picard group should be local quadratic entry locus varieties.

1. Preliminaries

$(*)$ *Setting, terminology and notation*

Throughout the paper we consider $X \subset \mathbb{P}^N$ an irreducible complex projective manifold of dimension $n \geqslant 1$. X is assumed to be non-degenerate and c denotes its codimension, so that $N = n + c$. We also suppose that X is scheme theoretically an intersection of m hypersurfaces of degrees $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$. It is implicitly assumed that m is minimal, i.e. none of the hypersurfaces contains the intersection of the others. We put $d := \sum_{i=1}^{c}(d_i - 1)$. $X \subset \mathbb{P}^N$ is called *quadratic* if it is scheme theoretically an intersection of quadrics (i.e. $d_1 = 2$). Note that this happens precisely when $d = c$.

For $x \in X$ we let $\mathbf{T}_x X$ denote the (affine) Zariski tangent space to X at x, and write $T_x X$ for its projective closure in $\mathbb{P}^N$. H denotes a hyperplane section (class) of X. As usual, K_X stands for the canonical class of X. Also, if $Y \subset X$ is a submanifold, we denote by $N_{Y/X}$ its normal bundle. For a vector bundle E, $\mathbb{P}(E)$ stands for its projectivized bundle, using Grothendieck's convention.

We let $SX \subset \mathbb{P}^N$ be the *secant variety* of X, that is the closure of the locus of secant lines. The *secant defect* of X is the (nonnegative) number $\delta = \delta(X) := 2n + 1 - \dim(SX)$. We say X is *secant defective* when $\delta > 0$.

A secant defective manifold $X \subset \mathbb{P}^N$ is called a *local quadratic entry locus* manifold, LQELM for short, if any two general points $x, x' \in X$ belong to a δ-dimensional quadric $Q_{x,x'} \subset X$. See [Ru, IR2] for a systematic study of these special secant defective manifolds.

$X \subset \mathbb{P}^N$ is *conic-connected* if any two general points $x, x' \in X$ are contained in some conic $C_{x,x'} \subset X$. Clearly, any LQELM is also conic-connected. Classification results for conic-connected manifolds are obtained in [IR], working in the general setting of rationally connected manifolds.

For a general point $x \in X$, we denote by $\mathcal{L}_x$ the (possibly empty) scheme of lines contained in X and passing through x. We say that $X \subset \mathbb{P}^N$ is *covered by lines* if $\mathcal{L}_x$ is not empty for $x \in X$ a general point. We refer the reader to [De, Ko] for standard useful facts about the deformation theory of rational curves; we shall use them implicitly in the simplest case, that is lines on X.

Recall that X is Fano if $-K_X$ is ample. The *index* of X, denoted by $i(X)$, is the largest integer j such that $-K_X = jA$ for some ample divisor A.

$(**)$ *The projective second fundamental form*

We recall some general results which are probably well known to the experts but for which we are unable to provide a proper reference.

There are several possible equivalent definitions of the projective second fundamental form $|II_{x,X}| \subseteq \mathbb{P}(S^2(\mathbf{T}_x X))$ of an irreducible projective variety $X \subset \mathbb{P}^N$ at a general point $x \in X$, see for example [IL, 3.2 and end of Section 3.5]. We use the one related to tangential projections, as in [IL, Remark 3.2.11].

Suppose $X \subset \mathbb{P}^N$ is non-degenerate, as always, let $x \in X$ be a general point and consider the projection from $T_x X$ onto a disjoint $\mathbb{P}^{c-1}$

$$\pi_x : X \dashrightarrow W_x \subseteq \mathbb{P}^{c-1}. \tag{1.1}$$

The map π_x is associated to the linear system of hyperplane sections cut out by hyperplanes containing $T_x X$, or equivalently by the hyperplane sections singular at x.

Let $\varphi : \mathrm{Bl}_x X \to X$ be the blow-up of X at x, let

$$E = \mathbb{P}((\mathbf{T}_x X)^*) = \mathbb{P}^{n-1} \subset \mathrm{Bl}_x X$$

be the exceptional divisor and let H be a hyperplane section of $X \subset \mathbb{P}^N$. The induced rational map $\widetilde{\pi}_x : \mathrm{Bl}_x X \dashrightarrow \mathbb{P}^{c-1}$ is defined as a rational map along E since $X \subset \mathbb{P}^N$ is not a linear space; see for example the argument in [E2, 2.1 (a)]. The restriction of $\widetilde{\pi}_x$ to E is given by a linear system in $|\varphi^*(H) - 2E|_{|E} \subseteq |-2E_{|E}| = |\mathcal{O}_{\mathbb{P}((\mathbf{T}_x X)^*)}(2)| = \mathbb{P}(S^2(\mathbf{T}_x X))$.

Definition 1.1. The *second fundamental form* $|II_{x,X}| \subseteq \mathbb{P}(S^2(\mathbf{T}_x X))$ of an irreducible non-degenerate variety $X \subset \mathbb{P}^N$ of dimension $n \geqslant 2$ at a general point $x \in X$ is the non-empty linear system of quadric hypersurfaces in $\mathbb{P}((\mathbf{T}_x X)^*)$ defining the restriction of $\widetilde{\pi}_x$ to E.

Clearly $\dim(|II_{x,X}|) \leqslant c-1$ and $\widetilde{\pi}_x(E) \subseteq W_x \subseteq \mathbb{P}^{c-1}$. From this point of view the base locus on E of the second fundamental form $|II_{x,X}|$ consists of *asymptotic directions*, i.e. of directions associated to lines having a contact of order at least three with X at x. For example, as we shall see, when $X \subset \mathbb{P}^N$ is quadratic, the base locus of the second fundamental form consists of points representing tangent lines contained in X and passing through x, so that it is exactly (even scheme theoretically) $\mathcal{L}_x$.

Proposition 1.2. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible non-degenerate variety of secant defect $\delta \geqslant 1$. Then $\dim(|II_{x,X}|) = c - 1$ for $x \in X$ a general point.*

Proof. Let notation be as above. It is sufficient to show that $\dim(\widetilde{\pi}_x(E)) = n - \delta$ because $\widetilde{\pi}_x(E) \subseteq W_x$ and $W_x \subseteq \mathbb{P}^{c-1}$ is a non-degenerate variety, whose dimension is $n - \delta$ by the Terracini Lemma.

Let $TX = \bigcup_{x \in X} T_x X$ be the tangential variety of X. The following formula holds

$$\dim(TX) = n + 1 + \dim(\widetilde{\pi}_x(E)), \tag{1.2}$$

see [Te] (or [GH, 5.6, 5.7] and [IL, Proposition 3.13.3] for a modern reference).

The variety $X \subset \mathbb{P}^N$ is smooth and secant defective, so that $TX = SX$ by a theorem of Fulton and Hansen, [FH]. Therefore $\dim(TX) = 2n + 1 - \delta$ and from (1.2) we get $\dim(\widetilde{\pi}_x(E)) = n - \delta$, as claimed. □

$(***)$ *Some standard exact sequences*

Let V be a complex vector space of dimension $N + 1$ such that $\mathbb{P}(V) = \mathbb{P}^N$. Consider the restriction of the Euler sequence on $\mathbb{P}^N$ to X

$$0 \to \Omega^1_{\mathbb{P}^N|X} \to V \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0, \tag{1.3}$$

and the exact sequence on X

$$0 \to N^*_{X/\mathbb{P}^N} \to \Omega^1_{\mathbb{P}^N|X} \to \Omega^1_X \to 0. \tag{1.4}$$

From these exact sequences we deduce

$$0 \to N^*_{X/\mathbb{P}^N}(1) \to V \otimes \mathcal{O}_X \to \mathcal{P}_X \to 0, \tag{1.5}$$

and

$$0 \to \Omega^1_X(1) \to \mathcal{P}_X \to \mathcal{O}_X(1) \to 0, \tag{1.6}$$

where $\mathcal{P}_X$ is the first jet bundle of $\mathcal{O}_X(1)$.

If $\mathbb{G}(n, N)$ denotes the Grassmannian of n-planes in $\mathbb{P}^N$, let $\gamma_X : X \to \mathbb{G}(n, N)$ be the Gauss map of X, associating to a point $x \in X$ the point of $\mathbb{G}(n, N)$ corresponding to the projective tangent space $T_x X$ to X at x. By Zak' s Theorem on Tangencies, the morphism γ_X is finite and birational, see [Za, I.2.8].

Let $\mathcal{U}$ be the universal quotient bundle on $\mathbb{G}(n, N)$, which is a locally free sheaf of rank $n + 1$. We have a natural surjection

$$V \otimes \mathcal{O}_{\mathbb{G}(n,N)} \to \mathcal{U},$$

inducing the surjection

$$V \otimes \mathcal{O}_X \to \gamma_X^*(\mathcal{U}).$$

Then it is easy to see that $\mathcal{P}_X = \gamma_X^*(\mathcal{U})$; so, for every closed point $x \in X$ we have $\mathbb{P}(\mathcal{P}_X \otimes k(x)) = T_x X \subset \mathbb{P}(V)$.

The above surjection gives an embedding, over X, of $\mathbb{P}(\mathcal{P}_X) \to X$ into $\mathbb{P}(V) \times X \to X$ in such a way that the restriction π_1 of the projection $\mathbb{P}^N \times X \to \mathbb{P}^N$ to $\mathbb{P}(\mathcal{P}_X)$ maps $\mathbb{P}(\mathcal{P}_X)$ onto $TX = \bigcup_{x \in X} T_x X$.

By (1.6) we get

$$\gamma_X^*(\det(\mathcal{U})) = \det(\mathcal{P}_X) \simeq \omega_X \otimes \mathcal{O}_X(n + 1).$$

2. MANIFOLDS COVERED BY LINES

Let $X \subset \mathbb{P}^N$ be as in $(*)$. The following examples of manifolds covered by lines are of relevance to us.

Examples 2.1. (1) $X \subset \mathbb{P}^N$ a Fano complete intersection with $i(X) \geqslant 2$.
(2) (Mori) $X \subset \mathbb{P}^N$ a Fano manifold with $\operatorname{Pic}(X) = \mathbb{Z}\langle H\rangle$ and $i(X) > \frac{n+1}{2}$.
(3) $X \subset \mathbb{P}^N$ a conic-connected manifold, different from the Veronese variety $v_2(\mathbb{P}^n)$ or one of its isomorphic projections.

Fix some irreducible component, say $\mathcal{F}$, of the Hilbert scheme of lines on $X \subset \mathbb{P}^N$, such that X is covered by the lines in $\mathcal{F}$. Put $a =: \deg(N_{\ell/X})$ where $[\ell] \in \mathcal{F}$. Note that $a \geqslant 0$ and $a = \dim(\mathcal{F}_x)$, where $x \in X$ is a general point and $\mathcal{F}_x = \{[\ell] \in \mathcal{F} \mid x \in \ell\}$. Moreover, we may view $\mathcal{F}_x$ as a closed subscheme of $\mathbb{P}((\mathbf{T}_x X)^*) \cong \mathbb{P}^{n-1}$.

When the dimension of $\mathcal{F}_x$ is large, the study of manifolds covered by lines is greatly simplified by the following two facts:

First, we may reduce, via a Mori contraction, to the case where the Picard group is cyclic; this is due to Beltrametti–Sommese–Wiśniewski, see [BSW]. Secondly, the variety $\mathcal{F}_x \subset \mathbb{P}^{n-1}$ inherits many of the good properties of $X \subset \mathbb{P}^N$; this is due to Hwang, see [Hw]. See [BI] for an application of these principles.

Theorem 2.2. *Assume $a \geqslant \frac{n-1}{2}$. Then the following results hold:*
(1) ([BSW]) *There is a Mori contraction, say $\operatorname{cont}_{\mathcal{F}} : X \to W$, of the lines from $\mathcal{F}$; let F denote a general fiber of $\operatorname{cont}_{\mathcal{F}}$ and let f be its dimension;*
(2) ([Wi]) $\operatorname{Pic}(F) = \mathbb{Z}\langle H_F\rangle$, $i(F) = a + 2$ *and F is covered by the lines from $\mathcal{F}$ contained in F;*
(3) ([Hw]) $\mathcal{F}_x \subseteq \mathbb{P}^{f-1}$ *is smooth irreducible non-degenerate. In particular, F has only one maximal irreducible covering family of lines.*

The following very useful result was proved in [BEL, Corollary 4]. It will play a crucial role in what follows.

Theorem 2.3 ([BEL])**.** *Let $X \subset \mathbb{P}^N$ be as in $(*)$. If*

$$-K_X = \mathcal{O}_X(n+1-d),$$

then $X \subset \mathbb{P}^N$ is a complete intersection of type $(d_1, \ldots, d_c)$.

Proof. Under our hypothesis

$$\begin{aligned} h^{n+1}(\mathcal{I}_X(d-n-1)) &= h^n(\mathcal{O}_X(d-n-1)) \\ &= h^0(\omega_X \otimes \mathcal{O}_X(n+1-d)) = h^0(\mathcal{O}_X) \neq 0. \end{aligned}$$

Thus $X \subset \mathbb{P}^N$ fails to be d-regular and we can apply [BEL, Corollary 4]. □

Now we can prove the following theorem, showing that $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ "behaves better" than $X \subset \mathbb{P}^N$. Indeed we can control the number of equations defining $\mathcal{L}_x$, as in the classical Example 2.1 (1), but without assuming X to be a complete intersection.

Theorem 2.4. *Let $X \subset \mathbb{P}^N$ be as in $(*)$. For $x \in X$ a general point, put $a = \dim(\mathcal{L}_x)$. Then the following results hold:*

(1) *If $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is nonempty, it is set theoretically defined by (at most) d equations; in particular, we have $a \geqslant n-1-d$.*
(2) *If X is quadratic and $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is nonempty, $\mathcal{L}_x$ is scheme theoretically defined by (at most) c quadratic equations.*
(3) *If $d \leqslant n-1$, then $\mathcal{L}_x \neq \emptyset$; assume moreover that $n \geqslant c+2$ if X is quadratic. Then*
   (a) *$X \subset \mathbb{P}^N$ is a Fano manifold with $\operatorname{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$ and $i(X) = a+2$;*
   (b) *the following conditions are equivalent:*
      (i) *$X \subset \mathbb{P}^N$ is a complete intersection;*
      (ii) *$\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a complete intersection of codimension d;*
      (iii) *$a = n-1-d$.*
(4) *Assume that $\operatorname{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$, $a \geqslant \frac{n-1}{2}$ and $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a complete intersection. Then X is conic-connected, $a \leqslant n-c-1$ and $n \geqslant 2c+1$.*

Proof. Let us take a closer look at the proof of the previous result given in [BEL]. Since $X \subset \mathbb{P}^N$ is scheme theoretically defined by equations of degree $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_m$, we can choose $f_i \in H^0(\mathbb{P}^N, \mathcal{I}_X(d_i))$, $i = 1, \ldots, c$, such that, letting $Q_i = V(f_i) \subset \mathbb{P}^N$, we obtain the complete intersection scheme

$$Y = Q_1 \cap \cdots \cap Q_c = X \cup X',$$

where X' (if nonempty) meets X in a divisor.

Thus for $x \in X$ general we have that a line l passing through x is contained in $X \subseteq Y$ if and only if l is contained in Y, that is $\mathcal{L}_x(X) \subset \mathbb{P}^{n-1}$ coincides set theoretically with $\mathcal{L}_x(Y) \subset \mathbb{P}^{n-1}$. Suppose, as recalled above, that

$$Y = V(f_1, \ldots, f_c) = X \cup X' \subset \mathbb{P}^N$$

with f_i homogeneous polynomials of degree d_i and let $x \in X$ be a general point. Without loss of generality we can suppose $x = (1 : 0 : \ldots : 0)$. Then in the affine space $\mathbb{A}^N$ defined by $x_0 \neq 0$, we have $f_i = f_i^1 + f_i^2 + \cdots + f_i^{d_i}$, with f_i^j homogeneous of degree j in the variables $(y_1, \ldots, y_N)$, where $y_l = \frac{x_l}{x_0}$ for every $l \geqslant 1$ and $x = (0, \ldots, 0)$. The equations of $\mathbb{P}^{n-1}$ inside $\mathbb{P}^{N-1} = \mathbb{P}((\mathbf{T}_x\mathbb{P}^N)^*)$ are exactly $f_1^1 = \cdots = f_c^1 = 0$ while the equations of $\mathcal{L}_x(Y)$ inside $\mathbb{P}^{n-1}$ are $f_1^2 = \cdots = f_1^{d_1} = \cdots = f_c^2 = \cdots = f_c^{d_c} = 0$, which are exactly $\sum_{i=1}^c (d_i - 1) = d$. In particular $\mathcal{L}_x \subset \mathbb{P}^{n-1}$, if not empty, is set theoretically defined by these equations, proving (1).

To prove (2), assume that X is quadratic. Keeping the notation above, we see that $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is scheme theoretically defined by the equations $f_1^2, \ldots, f_m^2$ (modulo the ideal generated by $f_1^1, \ldots, f_c^1$). But the same homogeneous quadratic equations define the affine scheme $X \cap \mathbf{T}_x X \subset \mathbf{T}_x X$. In particular, the pointed affine cone over $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ and the scheme $(X \cap \mathbf{T}_x X) \setminus \{x\}$ coincide. Consider now, as above, $Y = V(f_1, \ldots, f_c) = X \cup X'$. We have that the pointed affine cone over $\mathcal{L}_x(Y) \subset \mathbb{P}^{n-1}$ and the scheme $(Y \cap \mathbf{T}_x Y) \setminus \{x\}$ coincide and are scheme theoretically defined by (at most) c quadratic equations. But X and Y coincide in

a neighborhood of x, hence the pointed affine cones over $\mathcal{L}_x(X)$ and $\mathcal{L}_x(Y)$ also coincide. As $\mathcal{L}_x(X)$ is smooth, this implies that $\mathcal{L}_x(Y)$ is also smooth and so they coincide as schemes. This shows (2).

Under the hypothesis of (3), we deduce $a \geqslant n-1-d \geqslant 0$.

The assertion about the Picard group in (a) follows from the Barth–Larsen Theorem, see [BL]. Since $x \in X$ is general and since $\mathcal{L}_x \neq \emptyset$ by the above argument, for every line l passing through x we have

$$-K_X \cdot l = 2 + a \geqslant 2,$$

concluding the proof of (a).

To show (b), let us remark that by the previous discussion (i) implies (ii); (ii) implies (iii) is obvious. Theorem 2.3 shows that (iii) implies (i).

We now pass to part (4). Since $a \geqslant \frac{n-1}{2}$, $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is smooth irreducible and non-degenerate by Theorem 2.2 (3). As $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a non-degenerate complete intersection of dimension greater or equal to one less its codimension, it follows from the Terracini Lemma that $S\mathcal{L}_x = \mathbb{P}^{n-1}$, see e.g. [Ru2]. Now we may apply [HK, Theorem 3.14] to infer that X is conic-connected. In particular, we have $\delta(X) > 0$. From Proposition 1.2 it follows that $\dim |II_{x,X}| = c-1$. As $\mathcal{L}_x$ is contained in the base-locus of $|II_{x,X}|$, it follows that $\mathcal{L}_x$ is contained in at least c linearly independent quadrics. But, being a complete intersection, the number of such quadrics can not exceed the codimension of $\mathcal{L}_x$ in $\mathbb{P}^{n-1}$. This means that $c \leqslant n-1-a$, which is what we wanted. □

Next we investigate manifolds covered by lines when $a \geqslant n-c$. In this case we show that for each line from some covering family $\mathcal{F}$ having $\dim(\mathcal{F}_x) = a$, there is some hyperplane which is tangent to X along it; in particular X can not be a complete intersection!

We refer to the setting in ($***$). The following considerations are an elaboration of Lemma 1 from [Bu].

Consider a line $l \subset X$. Then $N_{l/X} \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{n-1})$ with $a_1 \leqslant a_2 \leqslant \cdots \leqslant a_{n-1} \leqslant 1$. Let $a(l) = \#\{i \mid a_i = 1\}$. If l passes through a general point $x \in X$, then $a_1 \geqslant 0$ and $\deg(N_{l/X}) = a(l)$.

For every line $l \subset X$ we shall consider the following linear spaces:

$$\Big\langle \bigcup_{x\in l} T_xX \Big\rangle, \tag{2.1}$$

which is the linear span of the union of the tangent spaces at points $x \in l$ and

$$\bigcap_{x\in l} T_xX. \tag{2.2}$$

By the discussion from ($***$), we also have the equality

$$\pi_1(\mathbb{P}((\mathcal{P}_X)_{|l})) = \bigcup_{x\in l} T_xX. \tag{2.3}$$

Since we shall be interested in $\bigcup_{x\in l} T_xX$ and in the dimension of its linear span in $\mathbb{P}^N$, we shall analyze $\mathcal{P}_{X|l}$, where for the moment $l \subset X$ is an arbitrary line.

From (1.5) we deduce that $\mathcal{P}_X$ and $\mathcal{P}_{X|l}$ are generated by global sections so that

$$\mathcal{P}_{X|l} \simeq \bigoplus_{j=1}^{n+1-b_0(l)} \mathcal{O}_{\mathbb{P}^1}(b_j(l)) \bigoplus \mathcal{O}_{\mathbb{P}^1}^{b_0(l)} \tag{2.4}$$

with $b_j(l) > 0$ for every $j = 1, \ldots, n+1-b_0(l)$.

Let $\Pi = \mathbb{P}^b = \mathbb{P}(U) \subset \mathbb{P}^N$. We have a surjection $V \to U$ inducing a surjection $V \otimes \mathcal{O}_X \to U \otimes \mathcal{O}_X$ and hence an inclusion, over X, $\mathbb{P}(U) \times X \subset \mathbb{P}(V) \times X$. Given a subvariety $Y \subseteq X$ we have that $\Pi \subset T_y X$ for every $y \in Y$ if and only if the natural surjection $V \otimes \mathcal{O}_Y \to U \otimes \mathcal{O}_Y$ factorizes through $V \otimes \mathcal{O}_Y \to \mathcal{P}_{X|Y}$, that is if and only if there exists a surjection $\mathcal{P}_{X|Y} \to U \otimes \mathcal{O}_Y$. Thus for a line $l \subset X$ we obtain that if $\Pi = \mathbb{P}^b \subset T_x X$ for every $x \in l$, then $b \leqslant b_0(l) - 1$.

Proposition 2.5. (cf. [Bu, Lemma1]) *Let notation be as above and let $l \subset X$ be a line passing through a general point of X. Then:*

(1)
$$\mathcal{P}_{X|l} \simeq \mathcal{O}_{\mathbb{P}^1}^{a(l)+2} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{n-1-a(l)}$$

and
$$\dim\left(\left\langle \bigcup_{x \in l} T_x X \right\rangle\right) = N - \mathrm{h}^0(N^*_{X/\mathbb{P}^N}(1)_{|l}).$$

In particular the variety $\bigcup_{x\in l} T_x X$ is isomorphic to a linear projection of a cone with vertex a linear space of dimension $a(l)+1$ over the Segre variety $\mathbb{P}^1 \times \mathbb{P}^{n-2-a(l)}$ so that
$$\dim\left(\left\langle \bigcup_{x \in l} T_x X \right\rangle\right) \leqslant \min\{N, 2n-1-a(l)\}.$$

(2)
$$\dim\left(\bigcap_{x \in l} T_x X\right) = a(l) + 1$$

and for $x, y \in l$ general we have
$$T_y \widetilde{C}_x = \bigcap_{z \in l} T_z X,$$

where $\widetilde{C}_x$ is the irreducible component of the locus of lines through x to which l belongs.

(3) $\mathrm{h}^0(N_{X/\mathbb{P}^N}(-1)_{|l}) = N - a(l) - 1$;

(4) *If $N \geqslant 2n-1-a(l)$, then $\mathrm{h}^0(N^*_{X/\mathbb{P}^N}(1)_{|l}) \geqslant N + a(l) - 2n + 1$ with equality holding if and only if $\dim(\langle \bigcup_{x\in l} T_x X\rangle) = 2n-1-a(l)$.*

Proof. From the exact sequence
$$0 \to \mathcal{O}_l(-1) \to \mathcal{P}^*_{X|l} \to T_X(-1)_{|l} \to 0$$

and from $h^0(T_X(-1)_{|l}) = a(l)+2$ we deduce $b_0(l) = h^0(\mathcal{P}^*_{X|l}) = a(l)+2$. Moreover

$$n+1-b_0(l) \leqslant \sum_{j=1}^{n+1-b_0(l)} b_j(l) = \deg(\mathcal{P}_{X|l}) = (K_X+(n+1)H)\cdot l$$
$$= -a(l)-2+n+1 = n+1-b_0(l)$$

so that $b_j(l)=1$ for every $j=1,\dots,n-1-a(l)$, proving the first assertion.

Let $\Pi = \bigcap_{x\in l} T_xX = \mathbb{P}^b$. By the previous analysis we deduce $b \leqslant a(l)+1$. If l passes through the general point $x\in X$, consider $\widetilde{C}_x$, the irreducible component of the locus of lines through x to which l belongs, which is a cone whose vertex contains x. We know that $\dim(\widetilde{C}_x) = a(l)+1$ and that for every $y \in l$ obviously $T_y\widetilde{C}_x \subset T_yX$. Since $T_y\widetilde{C}_x$ does not depend on $y\in l$ for $y \in l$ general, we deduce that, for $x,y \in l$ general, $T_y\widetilde{C}_x \subseteq \bigcap_{z\in l} T_zX$. Therefore $a(l)+1 \leqslant b$ which together with the previous inequality yields $b = a(l)+1$ and

$$T_y\widetilde{C}_x = \bigcap_{z\in l} T_zX \tag{2.5}$$

for $x,y\in l$ general points.

The other implications easily follow from the exact sequences in $(***)$ and their geometrical interpretation. $\square$

3. Defective manifolds

We begin by recalling the main known results about LQEL manifolds; complete proofs of the statements in the next theorem may be found in [Ru, IR, IR2, Fu].

Theorem 3.1. *Assume $X\subset\mathbb{P}^N$ is a LQEL manifold of secant defect δ. Then:*

(1) *X is a Fano rational manifold with $\mathrm{rkPic}(X)\leqslant 2$.*
(2) *If $\mathrm{rkPic}(X)=2$ then X is one of:*
  (a) *$\mathbb{P}^a\times\mathbb{P}^b$ in its Segre embedding, or*
  (b) *the hyperplane section of the above, or*
  (c) *the blowing-up of $\mathbb{P}^n$ with center a linear space L, embedded by the linear system of quadrics through L.*
(3) *If $\mathrm{rkPic}(X)=1$, $X\cong v_2(\mathbb{P}^n)$ or $\mathrm{Pic}(X)=\mathbb{Z}\langle H\rangle$ and $i(X)=\frac{n+\delta}{2}$.*
(4) *If $\delta\geqslant 3$ then $\mathcal{L}_x\subset\mathbb{P}^{n-1}$ is again a LQELM, $S\mathcal{L}_x=\mathbb{P}^{n-1}$, $\dim(\mathcal{L}_x)=\frac{n+\delta}{2}-2$, $\delta(\mathcal{L}_x)=\delta-2$. If $\delta\geqslant\frac{n}{2}$, a complete classification is obtained.*
(5) *X is a complete intersection if and only if X is a quadric $\mathbb{Q}^n (\delta=n)$.*
(6) *If $X\not\cong\mathbb{Q}^n$ then $\delta\leqslant\frac{n+8}{3}$; equality cases are classified.*

Conjecture 3.2. ([IR2]) *Assume X is a LQELM with $\mathrm{Pic}(X)\cong\mathbb{Z}\langle H\rangle$. Then X is obtained by linear sections and/or isomorphic projections from a rational homogeneous manifold, in its natural minimal embedding.*

We recall that rational homogeneous manifolds are well understood. In particular, those which are secant defective are known to be LQEL manifolds and are completely classified, see [Ka, Za]. They turn out to be quadratic manifolds and

moreover we have that $\delta \leqslant 8$ if X is not a quadric. So we expect any linearly normal LQEL manifold to be quadratic. Conversely, any quadratic secant defective manifold whose Koszul syzygies are generated by linear ones is a LQEL manifold by [Ve].

Next we consider the case of dual defective manifolds. Working in different settings, Mumford and Landman called the attention on these very special but intriguing class of embedded manifolds. They have since then been studied thoroughly by Ein in [E2, E1] and by Beltrametti–Fania–Sommese in [BFS]. See also [LS, Mu, Mu2].

For $X \subset \mathbb{P}^N$, let $X^* \subset \mathbb{P}^{N*}$ be the dual variety of X and let $\operatorname{def}(X) = N - 1 - \dim(X^*)$ be the *dual defect* of X.

For a hyperplane $H \subset \mathbb{P}^N$ we define the *contact locus* of H on X as

$$L = L(H) = \{x \in X \mid T_xX \subseteq H\} = \operatorname{Sing}(X \cap H) \subset X. \tag{3.1}$$

If $[H] \in X^*$ corresponds to a smooth point in X^*, then by Reflexivity $L(H) \simeq \mathbb{P}^{\operatorname{def}(X)}$ is an embedded linear subspace of $\mathbb{P}^N$ contained in X. Any line $l \subset L(H)$ with $[H] \in X^*$ is called a *contact line* on X. We recall a basic result on the geometry of contact lines proved by Ein in [E2, Section 2].

Theorem 3.3 ([E2])**.** *Let $k = \operatorname{def}(X) > 0$ and let $l \subset L(H)$ be a contact line with $[H] \in X^*$ general. Then $N_{l/X} \simeq \mathcal{O}_{\mathbb{P}^1}^{\frac{n-k}{2}} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\frac{n+k-2}{2}}$; in particular, n and k have the same parity, as first proved by Landman.*

Let $k = \operatorname{def}(X) > 0$ and let $x \in X$ be a general point. By Theorem 3.3 there exists a unique irreducible component of $\mathcal{L}_x$ of dimension $\frac{n+k-2}{2}$ containing a given general contact line. The union of all these irreducible components of $\mathcal{L}_x$ is called C_x. Since $\mathcal{L}_x$ is smooth and since $\frac{n+k-2}{2} \geqslant \frac{n-1}{2}$, there exists a unique irreducible component of $\mathcal{L}_x$ containing all the general contact lines passing through x, i.e. $C_x \subseteq \mathcal{L}_x$ is an irreducible component of $\mathcal{L}_x$ of dimension $\frac{n+k-2}{2}$ and hence an irreducible smooth subvariety of $\mathcal{L}_x$. Easy examples like the Segre varieties $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ show that in general $C_x \subsetneq \mathcal{L}_x$. For every line $[l] \in C_x$, we get $a(l) = \frac{n+k-2}{2}$.

Proposition 3.4. *Let $X \subset \mathbb{P}^N$ be as in $(*)$ with $\operatorname{def}(X) = k > 0$ and let $x \in X$ be a general point. If $k \geqslant n - 2c + 2$, then for a general line $[l] \in C_x$*

$$\dim\big(\big\langle \bigcup_{x \in l} T_xX \big\rangle\big) = \frac{3n-k}{2} \quad \textit{and} \quad N_{X/\mathbb{P}^N}(-1)_{|l} \simeq \mathcal{O}_{\mathbb{P}^1}^{\frac{k+2c-n}{2}} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\frac{n-k}{2}}.$$

Moreover, every line $[l] \in C_x$ is a contact line and for a general $[l] \in C_x$ every hyperplane H such that $l \subset L(H)$ represents a smooth point of X^*. In particular through a general line $[l] \in C_x \subset \mathbb{P}^{n-1}$ there passes a linearly embedded $\mathbb{P}^{k-1}$ contained in C_x, corresponding to lines in $L(H)$ passing through x.

Proof. If $[l] \in C_x$, then $a(l) = \frac{n+k-2}{2}$ so that the hypothesis and Proposition 2.5 imply

$$\dim\big(\big\langle \bigcup_{x \in l} T_xX \big\rangle\big) \leqslant 2n - 1 - a(l) \leqslant N - 1. \tag{3.2}$$

Thus every line $[l] \in C_x$ is contained in the contact locus of at least one hyperplane H with $[H] \in (T_xX)^*$.

Let $X_x^* = \{[H] \in X^* \mid T_xX \subseteq H\} = (T_xX)^* \simeq \mathbb{P}^{c-1}$. Put

$$Z_x = \{([l],[H]) \mid l \subset L(H)\} \subset C_x \times X_x^*,$$

where $p_x : Z_x \to C_x$ is the restriction of the first projection and $q_x : Z_x \to X_x^*$ the restriction of the second projection. For every $[H] \in \mathrm{Sm}(X^*) \cap X_x^*$ we have $q_x^{-1}([H]) = \mathbb{P}^{k-1}$, where $\mathrm{Sm}(X^*)$ denotes the smooth locus of X^*. Thus q_x is surjective and every irreducible component of Z_x dominating X_x^* has dimension $k+c-2$. By (3.2) also p_x is surjective and by definition of Z_x and C_x every irreducible component of Z_x dominating C_x dominates X_x^* and vice versa. Let $U = q_x^{-1}(\mathrm{Sm}(X^*) \cap X_x^*)$. Since p_x is proper and surjective, there exists an open subset $V \subseteq C_x$ such that $p_x^{-1}(V) \subseteq U$. Thus for every $[l] \in V$, every hyperplane H such that $l \subset L(H)$ is a smooth point of X^* and through $[l]$ there passes a linear embedded $\mathbb{P}^{k-1} \subset \mathbb{P}^{n-1}$ contained in C_x.

Let $[l] \in C_x$ be an arbitrary line and let $F_{[l]} = p_x^{-1}([l])$. By definition $F_{[l]} = (\langle \bigcup_{x \in l} T_xX \rangle)^* = \mathbb{P}^{N-b(l)-1}$, where $b(l) = \dim(\langle \bigcup_{x\in l} T_xX\rangle)$. Thus for a general $[l] \in C_x$

$$\dim(F_{[l]}) = k + c - 2 - \frac{n+k-2}{2} = \frac{k+2c-2-n}{2}, \tag{3.3}$$

and

$$n + c - b(l) - 1 = N - b(l) - 1 = \dim(F_{[l]}) = \frac{k+2c-2-n}{2}, \tag{3.4}$$

yielding $b(l) = \dim(\langle \bigcup_{x\in l} T_xX\rangle) = \frac{3n-k}{2} = 2n - 1 - a(l)$.

The locally free sheaf $N_{X/\mathbb{P}^N}(-1)_{|l}$ has rank c and is generated by global sections, so that

$$N_{X/\mathbb{P}^N}(-1)_{|l} \simeq \bigoplus_{j=1}^{c} O_{\mathbb{P}^1}(d_j),$$

with $d_j \geqslant 0$. Let $d_0(l) = \#\{i \mid d_i = 0\}$ and let $d_1 \leqslant \cdots \leqslant d_c$. The exact sequences

$$0 \to N_{L/X}(-1)_{|l} \to \mathcal{O}_l^{N-k} \to N_{X/\mathbb{P}^N}(-1)_{|l} \to 0$$

and

$$0 \to \mathcal{O}_l^{k-1} \to N_{l/X}(-1) \to N_{L/X}(-1)_{|l} \to 0$$

together with Theorem 3.3 yield

$$\frac{n-k}{2} = -\deg(N_{l/X}(-1)) = \deg(N_{X/\mathbb{P}^N}(-1)_{|l}) = \sum_{j=d_0(l)+1}^{c} d_j \geqslant c - d_0(l).$$

Thus $d_0(l) \geqslant c - \frac{n-k}{2} = \frac{k+2c-n}{2}$ with equality holding if and only if $d_r = 1$ for every $r = d_0(l)+1, \ldots, c$.

By definition of $d_0(l)$ and by Proposition 2.5 part (4) we get

$$d_0(l) = \mathrm{h}^0(N^*_{X/\mathbb{P}^N}(1)_{|l}) = N + a(l) - 2n + 1 = \frac{k+2c-n}{2}, \tag{3.5}$$

so that $N_{X/\mathbb{P}^N}(-1)_{|l} \simeq \mathcal{O}_{\mathbb{P}^1}^{\frac{k+2c-n}{2}} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\frac{n-k}{2}}$, as claimed. □

Lemma 3.5. *Let $X \subset \mathbb{P}^N$ be as in $(*)$ with $\operatorname{def}(X) = k > 0$. If $k \geqslant \frac{n}{3}$, then $k \geqslant n - 2c + 2$.*

Proof. By Zak's Theorem on Tangencies, $k \leqslant c - 1$. Then

$$n \leqslant 3k \leqslant k + 2c - 2,$$

concluding the proof. □

Let us recall that $X \subset \mathbb{P}^N$ is an r*-scroll*, if $X \simeq \mathbb{P}(E)$ where E is a rank $r + 1$ vector bundle over some manifold W and the fibers of the projection $\pi : \mathbb{P}(E) \to W$ are linearly embedded in $\mathbb{P}^N$. When $r > \dim(W)$, the scroll X is dual defective and its defect equals $r - \dim(W)$. The next result characterizes scrolls among all dual defective manifolds.

Theorem 3.6. *Let $X \subset \mathbb{P}^N$ be as in $(*)$ with $\operatorname{def}(X) = k > 0$. Let $C_x \subset \mathbb{P}^{n-1}$ be as above and put $\mathbb{P}^m \simeq T \subseteq \mathbb{P}^{n-1}$ to be the linear span of C_x in $\mathbb{P}^{n-1}$. The following conditions are equivalent:*

(i) *X is an $\frac{n+k}{2}$-scroll over a manifold of dimension $\frac{n-k}{2}$;*
(ii) *$\dim(C_x) > 2\operatorname{codim}_T(C_x)$ (or equivalently $k > \frac{4m+6-3n}{3}$).*

Proof. We only have to prove that (ii) implies (i). Since $n \equiv k$ modulo 2, it follows that $\dim(C_x) = \frac{n+k-2}{2} \geqslant [\frac{n}{2}]$. In particular the lines in C_x generate an extremal ray of X by Theorem 2.2 (1). Let $\varphi : X \to W$ be the contraction of this ray and let F be a general fiber of φ. Then $F \subset \mathbb{P}^N$ is a smooth irreducible projective variety such that $\operatorname{Pic}(F) \simeq \mathbb{Z}\langle H_F\rangle$. Let $f = \dim(F)$ and let $\langle F\rangle$ be the linear span of F in $\mathbb{P}^N$. Then $C_x \subseteq \mathbb{P}((\mathbf{T}_xF)^*) = \mathbb{P}^{f-1}$ is smooth irreducible non-degenerate by Theorem 2.2 (3). Thus $m = f - 1$. Moreover, by [BFS, Theorem (1.2)] we have $\operatorname{def}(F) = k(F) = k + n - f$ so that the hypothesis in (ii) yields $k(F) > \frac{f+2}{3}$. By Lemma 3.5 and Proposition 3.4 every line in C_x is a contact line for $F \subseteq \langle F\rangle$ and $C_x \subseteq \mathbb{P}^{f-1}$ is covered by linear spaces of dimension $k(F) - 1 > [\frac{\dim(C_x)}{2}]$. From [Sa] (see also [BI] for a simple proof in the spirit of the present paper) it follows that $C_x \subseteq \mathbb{P}((\mathbf{T}_xF)^*) = \mathbb{P}^{f-1}$ is a scroll. The condition in (ii) and the Barth-Larsen Theorem, see [BL], give $\operatorname{Pic}(C_x) \simeq \mathbb{Z}\langle H_{C_x}\rangle$. So we get $C_x = \mathbb{P}^{f-1}$, hence $F = \mathbb{P}^f$ and $k(F) = f$. Therefore $f = \frac{n+k}{2} > \frac{n}{2}$ and $X \subset \mathbb{P}^N$ is a scroll by [E1, Theorem 1.7]. □

Corollary 3.7. *Let $X \subset \mathbb{P}^N$ be as in $(*)$, with $\operatorname{def}(X) = k > 0$. Let $\varphi : X \to W$ be the contraction whose existence is ensured by Theorem 2.2 (1). Assume that X is not a scroll. Then*

$$k \leqslant \frac{n + 2 - 4\dim(W)}{3} \leqslant \frac{n+2}{3}.$$

Moreover:

(1) *$k = \frac{n+2}{3}$ if and only if $N = 15$, $n = 10$ and $X \subset \mathbb{P}^{15}$ is projectively equivalent to the 10-dimensional spinorial variety $S^{10} \subset \mathbb{P}^{15}$.*

(2) *If* $\dim(W) > 0$*, then* $k = \frac{n+2-4\dim(W)}{3}$ *if and only if* $\dim(W) = n - 10 \leqslant 3$ *and* $\varphi : X \to W$ *is a fibration such that the general fiber* $F \subset \langle F\rangle \subset \mathbb{P}^N$ *is isomorphic to* $S^{10} \subset \mathbb{P}^{15} \subset \mathbb{P}^N$.

Proof. Keeping the notation from the preceding theorem, we have $k \leqslant \frac{4m+6-3n}{3}$, $m = f - 1$ and $f = n - \dim(W)$. So we may assume that $k = \frac{n+2}{3}$. Then $\dim(C_x) = \frac{2(n-1)}{3}$. By Lemma 3.5 and Proposition 3.4 $C_x \subset \mathbb{P}^{n-1}$ is covered by linear spaces of dimension $k - 1 = \frac{n-1}{3} = \frac{\dim(C_x)}{2}$. Then by [Sa] and [NO] the variety $C_x \subset \mathbb{P}^{n-1}$ is one of the following:

a) a scroll;
b) a quadric hypersurface of even dimension;
c) $\mathbb{G}(1, r)$ Plücker embedded with $r \geqslant 4$, or one of its isomorphic projections.

If $n < 7$, then $n = 4$ and $k = n - 2$ so that $X \subset \mathbb{P}^n$ would be a scroll over a curve by [E2, Theorem 3.2], contradicting our assumption. If $C_x \subset \mathbb{P}^{n-1}$ is a (quadric) hypersurface, then $\frac{2(n-1)}{3} = n - 2$ yields $n = 4$, obtaining once again a contradiction. Thus we can suppose $n \geqslant 7$, so that $\mathrm{Pic}(C_x) = \mathbb{Z}\langle H\rangle$ by the Barth-Larsen Theorem. Therefore we are necessarily in case c). Since the secant defect of $\mathbb{G}(1, r)$ is four, we get: $n - 1 \geqslant 2\dim(C_x) + 1 - \delta(C_x) = \frac{4n-13}{3}$, yielding $n \leqslant 10$. On the other hand, we have $\dim(C_x) = 2(r - 1) \geqslant 6$, so that $n \geqslant 10$. Therefore $n = 10$, $k = 4$ and $i(X) = 8$; moreover, the proof of the previous theorem shows that $F = X$, i.e. $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$. The conclusion of part (1) follows by Mukai's classification of prime Fano manifolds of index $n - 2$, see [Muk].

To prove (2), assume that $k = \frac{n+2-4\dim(W)}{3}$. By [BFS, Theorem (1.2)]

$$k(F) = \frac{n + 2 - 4\dim(W)}{3} + \dim(W) = \frac{f + 2}{3},$$

so that by the first part $f = 10 = n - \dim(W)$ and $F \subset \langle F\rangle \subset \mathbb{P}^N$ is isomorphic to $S^{10} \subset \mathbb{P}^{15}$. □

Remark 3.8. The preceding corollary improves one of the main results in [E1] stating that if $k > 0$ and X is not a scroll, we have $k \leqslant \frac{n-2}{2}$, with equality only if X is projectively isomorphic either to $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ or to $S^{10} \subset \mathbb{P}^{15}$. The next corollary is the main result of [E2], proved in a different manner. Another proof, based on the theory of LQEL manifolds, may be found in [IR2].

Corollary 3.9. ([E2, Theorem 4.5]) *Let* $X \subset \mathbb{P}^N$ *be a smooth irreducible non-degenerate variety such that* $\dim(X) = \dim(X^*)$*. If* $n \leqslant 2c$*, then* X *is projectively equivalent to one of the following:*

(a) *a smooth hypersurface* $X \subset \mathbb{P}^{n+1}$, $n = 1, 2$;
(b) *a Segre variety* $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$;
(c) *the Plücker embedding* $\mathbb{G}(1, 4) \subset \mathbb{P}^9$;
(d) *the* 10*-dimensional spinorial variety* $S^{10} \subset \mathbb{P}^{15}$.

Proof. By hypothesis $k = c-1$. Clearly $k = 0$ if and only if $X \subset \mathbb{P}^{n+1}$ is a hypersurface, giving case (a). From now on we suppose $k > 0$. By [E2, Theorem 3.2]

$k = n-2$ and $N = 2n-1$ if and only if we are in case (b); see also [E2, Theorem 3.3, c)]. If $k \leqslant n-3$, then $n \geqslant c+2$ so that by the Barth–Larsen Theorem we deduce $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$. Hence we can assume that $X \subset \mathbb{P}^N$ is not a scroll. Then by Corollary 3.7 either we are in case (d) or we have $\frac{n+2}{3} > k = c-1 \geqslant \frac{n-2}{2}$. Taking into account that we have $n \equiv k$ modulo 2, this leads to only one more case, namely $n = 6, k = 2$. So we also have $i(X) = 5, c = 3, N = 9$ and we are in case (c) by Fujita's classification of del Pezzo manifolds, see [Fuj]. □

Conjecture 3.10. ([IR2]) *Any dual defective manifold with cyclic Picard group is a LQELM.*

Theorem (1.2) from [BFS], based on Theorem 2.2 (1), reduces the study of dual defective manifolds to the case when $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$. Note that $k \leqslant 4$ in all known examples other than scrolls.

Proposition 3.11. *Conjecture 3.10 for a dual defective manifold $X \subset \mathbb{P}^N$ as in* $(*)$ *with dual defect k is equivalent to the conjunction of the following two assertions:*

(1) *X is conic-connected, and*
(2) $k \geqslant n-c-1$.

Proof. If X is a LQELM, of defect δ, $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$ and X is dual defective of defect k, it follows that $\delta = k+2$. Indeed, we have

$$\frac{n+\delta-4}{2} = \dim(\mathcal{L}_x) = \frac{n+k-2}{2}.$$

Assume now that X is dual defective and (1) and (2) hold. Since X is conic-connected, it follows from [IR2, Proposition 3.2] that X is a LQELM if $i(X) \geqslant \frac{n+\delta}{2}$. Replacing X by one of its isomorphic projections allows us to assume that $SX = \mathbb{P}^N$, hence $\delta = n-c+1$. Therefore the inequality in (2) becomes $k+2 \geqslant \delta$. But we have

$$i(X) = \dim(\mathcal{L}_x) + 2 = \frac{n+k+2}{2} \geqslant \frac{n+\delta}{2},$$

so that X is a LQELM.

Conversely, if $\mathrm{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$ and X is (dual defective and) a LQELM, we trivially have condition (1). As for (2), by the above we have $k = \delta - 2$; so condition (2) becomes $\delta \geqslant n-c+1$, which is obvious. □

Remarks 3.12. (1) Condition (2) above implies the inequality $k \geqslant n-2c+2$, which is the hypothesis in Proposition 3.4 (use that $n \equiv k$ modulo 2).

(2) By Zak's Theorem on Tangencies $k \leqslant c-1$, which combined with the inequality in (2) yields $n \leqslant 2c$ for dual defective manifolds. Note that this was a hypothesis in Corollary 3.9.
(3) Knowing that (2) holds would lead to a much simpler proof of Corollary 3.7, without making use of the elaborate results from [Sa, NO].
(4) If X is both dual defective and a LQELM, we have seen that $\delta = k+2$; therefore, the upper bounds on δ in Theorem 3.1, and on k in Corollary 3.7, are the same. They express the condition that

$$\dim(\mathcal{L}_x) \leqslant 2\,\mathrm{codim}(\mathcal{L}_x, \mathbb{P}^{n-1}).$$

Moreover, quadrics are the only complete intersections that are LQELMs, while linear spaces (generating scrolls) are the only complete intersections among dual defective manifolds. This naturally leads us to the next section.

4. Around the Hartshorne Conjecture; the quadratic case

In [Ha], Hartshorne made his now famous conjecture:

Conjecture 4.1 (HC). *If $X \subset \mathbb{P}^N$ is as in $(*)$ and $n \geqslant 2c+1$ then X is a complete intersection.*

This is still widely open, even for $c = 2$. We would like to state explicitly several weaker versions.

Conjecture 4.2 (HCF). *Assume that $n \geqslant 2c + 1$ and X is Fano; then X is a complete intersection.*

Conjecture 4.3 (HCL). *Assume that $X \subset \mathbb{P}^N$ is covered by lines with $\dim(\mathcal{L}_x) \geqslant \frac{n-1}{2}$ and let $T = \langle \mathcal{L}_x \rangle \subseteq \mathbb{P}^{n-1}$. If $\dim(\mathcal{L}_x) > 2\operatorname{codim}_T(\mathcal{L}_x)$, then $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a complete intersection.*

Remarks 4.4.
(1) When $n \geqslant 2c+1$, by the Barth–Larsen Theorem $\operatorname{Pic}(X) \cong \mathbb{Z}\langle H \rangle$. In particular $K_X = bH$ for some integer b. So, saying that X is Fano means exactly that $b < 0$; this happens, for instance, if X is covered by lines.
(2) The (HCF) holds when $c = 2$, see [BC].
(3) By Theorem 2.2, the (HCL) concerns Fano manifolds $X \subset \mathbb{P}^N$ with $\operatorname{Pic}(X) \simeq \mathbb{Z}\langle H \rangle$ and of index $i(X) \geqslant \frac{n+3}{2}$. Dual defective and LQEL manifolds satisfy the (HCL), by Theorem 3.6 and Theorem 3.1.
(4) Manifolds of (very) small degree are known to be complete intersections, cf. [Ba]. The (HCF) would yield the following optimal result, see [Io].

Conjecture 4.5. *If $n \geqslant degree(X) + 1$, then X is a complete intersection, unless it is projectively equivalent to $\mathbb{G}(1,4) \subset \mathbb{P}^9$.*

Note that the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$ has degree n and is not a complete intersection if $n \geqslant 3$.

Prime Fano manifolds of high index tend to be complete intersections. In the next proposition we propose such bounds on the index and show how they would follow from the (HC). However, we expect this kind of result to be easier to prove than the general (HC).

Proposition 4.6.
(1) *Let $X \subset \mathbb{P}^N$ be a Fano manifold with $\operatorname{Pic}(X) \simeq \mathbb{Z}\langle H \rangle$ and of index $i(X) \geqslant \frac{2n+5}{3}$. If the* (HCL) *and the* (HCF) *are true, then X is a complete intersection.*
(2) *The same conclusion holds assuming only the* (HCF)*, but asking instead that $i(X) \geqslant \frac{3(n+1)}{4}$.*

Proof. (1) Since $i(X) > \frac{n+1}{2}$ we get that X is covered by lines by Example 2.1 (2). We have $\dim(\mathcal{L}_x) = i(X) - 2 \geqslant \frac{2n-1}{3} \geqslant \frac{n-1}{2}$, hence $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is smooth

irreducible non-degenerate by Theorem 2.2 (3). Moreover, we have $\dim(\mathcal{L}_x) \geqslant 2\operatorname{codim}(\mathcal{L}_x, \mathbb{P}^{n-1}) + 1$. Thus $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a complete intersection by the (HCL). Next we may apply Theorem 2.4 (4) to infer that $n \geqslant 2c+1$ and then use the (HCF).

(2) Since $i(X) > \frac{2n}{3}$ we have that X is conic-connected by [HK]. From [IR2, Proposition 3.2] we infer

$$\frac{3(n+1)}{4} \leqslant i(X) \leqslant \frac{n+\delta}{2}.$$

This yields $\delta > \frac{n}{2}$ so that $SX = \mathbb{P}^N$ by Zak's Linear Normality Theorem. Therefore $\delta = n - c + 1$ and

$$\frac{3(n+1)}{4} \leqslant i(X) \leqslant \frac{n+(n-c+1)}{2}$$

implies $n \geqslant 2c+1$, so the (HCF) applies. □

Note that for complete intersections $X \subset \mathbb{P}^N$, $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is also a complete intersection. We believe that for manifolds covered by lines the converse should also hold.

Conjecture 4.7. *If $X \subset \mathbb{P}^N$ is covered by lines and $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a (say smooth irreducible non-degenerate) complete intersection, then X is a complete intersection too.*

Theorem 2.4 (4) shows that the above conjecture follows from the (HCF), at least when $\dim(\mathcal{L}_x) \geqslant \frac{n-1}{2}$.

Mumford in his seminal series of lectures [Mum] called the attention to the fact that many interesting embedded manifolds are scheme theoretically defined by quadratic equations.

As a special case of the results in [BEL], if X is quadratic, we have:

(1) If $n \geqslant c-1$ then X is projectively normal;
(2) If $n \geqslant c$ then X is projectively Cohen–Macaulay.

Our main results in this section are:

Theorem 4.8. *Assume that $X \subset \mathbb{P}^N$ is as in $(*)$ and X is quadratic.*

(1) *If $n \geqslant c+1$ then X is covered by lines. Moreover, $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is scheme theoretically defined by c independent quadratic equations.*
(2) *If $n \geqslant c+2$ then X is a Fano manifold with $\operatorname{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$. Moreover, the following conditions are equivalent:*
   (i) *$X \subset \mathbb{P}^N$ is a complete intersection;*
   (ii) *$\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is a complete intersection;*
   (iii) *$\dim(\mathcal{L}_x) = n-1-c$;*
   (iv) *$N_{X/\mathbb{P}^N}(-1)$ is ample.*
(3) (HC) *If $n \geqslant 2c+1$ then X is a complete intersection.*
(4) *If $\operatorname{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$ and X is Fano of index $i(X) \geqslant \frac{2n+5}{3}$, then X is a complete intersection.*

Proof. We use the notation in $(*)$. Moreover, denote by $a := \dim(\mathcal{L}_x)$. Since X is quadratic, we have $d = c$.

From Theorem 2.4 (2) and (3) we deduce the first part in (1) and (2). It also follows that $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is defined scheme theoretically by at most c quadratic equations. Since $n \geqslant c+1$, we must have $\delta \geqslant n-c+1 > 0$. From Proposition 1.2 we infer that the quadratic equations defining $\mathcal{L}_x$ are independent, proving (1). This also shows that, when $\mathcal{L}_x$ is a complete intersection, it has codimension c in $\mathbb{P}^{n-1}$. From Theorem 2.4 (3) the equivalence between conditions (i), (ii) and (iii) is now clear.

Assume that $N_{X/\mathbb{P}^N}(-1)$ is ample. We claim that $a \leqslant n-1-c$. Indeed, if $a \geqslant n-c$, from Proposition 2.5 we deduce that $\mathrm{h}^0(N^*_{X/\mathbb{P}^N}(1)_{|l}) > 0$. Therefore the restriction $N_{X/\mathbb{P}^N}(-1)|_l$ can not be ample and the claim is proved. Combined with the first part of Theorem 2.4, this shows that $a = n-1-c$, so that X is a complete intersection. The equivalence of the four conditions in (2) is thus proved.

To prove part (3), assume that $n \geqslant 2c+1$. We first observe that by Theorem 2.4 (1) we have $a \geqslant n-1-c \geqslant \frac{n-1}{2}$. Next, since $n \geqslant 2c+1$, $\operatorname{Pic}(X) \simeq \mathbb{Z}\langle H\rangle$ by [BL]. From Theorem 2.2(3) $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is smooth irreducible and non-degenerate. Being defined scheme theoretically by $c \leqslant \frac{n-1}{2}$ equations, it is a complete intersection by the result of Faltings [Fa]. By the previous point X is a complete intersection, too.

Finally, part (4) follows from part (3) and Proposition 4.6. □

Theorem 4.9. *Assume that $X \subset \mathbb{P}^N$ is as in $(*)$ and X is quadratic. If $n = 2c$ and X is not a complete intersection, then it is projectively equivalent to one of the following:*

(a) $\mathbb{G}(1,4) \subset \mathbb{P}^9$, *or*
(b) $S^{10} \subset \mathbb{P}^{15}$.

Proof. We need a refinement of Faltings' Theorem from [Fa], due to Netsvetaev, see [Ne]. Since we assumed that X is not a complete intersection, we have $a \geqslant n-c = \frac{n}{2} \geqslant \frac{n-1}{2}$, so $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ is smooth irreducible non-degenerate by Theorem 2.2 (3). Moreover, by the previous theorem, $\mathcal{L}_x$ is not a complete intersection in $\mathbb{P}^{n-1}$, but it is scheme theoretically defined by c independent quadratic equations. Netsvetaev's Theorem from [Ne] applied to $\mathcal{L}_x \subset \mathbb{P}^{n-1}$ gives the inequalities
$$\frac{3n-6}{4} \leqslant a < \frac{3n-5}{4}.$$
We may assume $n \geqslant 6$ (otherwise $\mathcal{L}_x$ is a complete intersection), so we deduce that $n = 4r+2$ and $a = 3r$, for a suitable r. If $r > 2$ (equivalently $n > 10$), we would have $a \geqslant 2(n-1-a)+1$ and $\mathcal{L}_x$ would be a complete intersection by Theorem 4.8 (3). In conclusion $n = 6$ or $n = 10$. In the first case $i(X) = a+2 = 5$ and we get case (a) by the classification of del Pezzo manifolds, see [Fuj]. In the second case, $i(X) = 8$, leading to case (b) by [Muk]. □

Remarks 4.10. (1) If $X \subset \mathbb{P}^{\frac{3n}{2}}$ is not a complete intersection, it is called a Hartshorne manifold. $\mathbb{G}(1,4) \subset \mathbb{P}^9$ and $S^{10} \subset \mathbb{P}^{15}$ are such examples, the

first one being due to Hartshorne himself. So, for quadratic manifolds we proved not only that the (HC) holds, but also that the above examples are the only Hartshorne manifolds. This shows, once again, how mysterious the general case of the (HC) remains!

(2) Working with local differential geometric techniques, as in [GH, IL], Landsberg [La] proved that a (possibly singular) quadratic variety satisfying $n \geqslant 3c + b - 1$ is a complete intersection, where $b = \dim(\mathrm{Sing}(X))$.

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